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GREEN'S FUNCTION APPROACH TO THE
MANY BODY PROBLEM

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
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MICHAEL REVZEN

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A B S T R A C T

The Green's function approach to the temperature-dependent many body problem is reviewed in some detail (Chapter II). A compact formulation of the problem is proposed which emphasizes the well-known parallelism between the roles of the time and a certain temperature-variable in the temperature-dependent many body problem. This formulation is then used to derive in a new way some formulae previously established in the literature (Chapter III). The energy-momentum representations for the Green's functions are then obtained, and the parallelism which was mentioned above is discussed. A result of this discussion is to show that the parallelism, though inexact, is far-reaching. In this connection it is shown that the Green's function pertaining to the temperature variable can be split into two parts which can be associated unambiguously with the retarded and advanced parts of Green's functions pertaining to the time (Chapter IV). In the last chapter the main results of this thesis are discussed in the context of the recent literature and some suggestions are made for further investigations.

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CHAPTER I: INTRODUCTION

In dealing with the many body problem (MBP) one is concerned with an idealized physical situation, viz., one investigates the properties of a system composed of N particles in a volume V where both N and V are infinite but their ratio (N/V) is finite. It is clear from the outset that for a system with a large number of interacting particles one cannot hope to solve the Schrodinger equation

$$H\psi = E\psi$$

exactly, i.e., one cannot expect to get a solution for ψ in closed form. Furthermore, we are not really interested in the detailed solution; what is of interest to us in the many body problem is to explain the gross properties of the system under consideration in terms of some fundamental properties of the particles which make up the system. To be specific the fundamental properties that are of interest here are the mutual interactions among particles and the statistics obeyed by the particles.

As a result of the mathematical complexity of the problem it is customary to use models to describe a system composed of a large number of interacting particles. Thus, the fission of nuclei is pictured as the fission of a drop of liquid. Another process for which a liquid drop model of nuclei is useful is the photoemission of nucleons from heavy nuclei. As is to be expected, this model has its limitations; for example, the photoemission of high energy nucleons from heavy nuclei cannot be explained in terms of the liquid drop model.

The purpose of a satisfactory many body theory in the case of zero

temperature is to obtain a convenient expression for the ground state of the system in terms of the fundamental properties of the particles which were specified above. Actually, when one wishes to calculate with this expression the characteristics of a given process, one must resort to approximations. These approximations may be different for different processes. In executing these approximations, one should be in a position to determine the region of validity of a given approximation. Since each approximation corresponds to a model, one can estimate from the outset the validity of a model for a particular process. For example, the Brueckner theory was shown by Goldstone (1957) to take into account a certain class of terms in the general expression for the ground state energy of nuclear matter. These terms were shown by Brueckner to justify the single particle model for such a system. Here we have an example of a many body theory which justifies a known model for nuclear matter; moreover, in this example the theory actually enriches the model inasmuch as it shows that the appropriate single particle^{potential} _{Λ} is velocity dependent.

The considerable progress in many body theory which has occurred in recent years is due to a large extent to the realization that the arsenal of field theory can be used in an attack on the MBP. This realization is usually attributed to Goldstone (1957) who used the techniques of field theory employing the so-called interaction picture. However, when applied to the MBP, field theory techniques employing the Green's function formalism possess a distinct advantage. This is primarily for two reasons: (a) In the Green's function formalism one considers only

the so-called linked cluster diagrams. These diagrams correspond to processes wherein particles are excited from the ground state, and after going through a variety of intermediate states excluding the ground state, return to the ground state. In the alternative approach using the interaction picture, one obtains terms which correspond to processes wherein particles return to the ground state before the end of the process ("unlinked clusters"). Although we expect that the terms corresponding to unlinked clusters can be eliminated, the procedure of proving this elimination is not rigorous (Goldstone, 1957; Bloch, 1958) (b) The Green's function formulation is concise and elegant. For example, in this dissertation the whole problem has been formulated without recourse to diagrammatic techniques. At this juncture we would like to quote the motivation given by Klein and Prange (1958) for introducing the Green's function formalism to the MBP: "The formal development of the subject (MBP) has been marked by an increasing reliance on the method of second quantization, not at all surprising in view of the fact that we are dealing from one standpoint with a class of particularly simple field theories. What has seemed surprising to the authors is that detailed treatments so far published have for the most part been either special to the problem at hand or else have taken over general techniques of quantum field theory which date from the 'early part' of the last decade of profound development of the methodology."

In this thesis we concern ourselves with the temperature-dependent many body problem (T - dependent MBP). In the T-dependent MBP we are interested in the function $\rho(E, N)$ which gives the density of

levels as a function of the energy and the number of particles per unit volume. A related function, with which we will deal directly, is the grand partition function Z . This latter function is defined by

$$Z = \text{Tr. } Z_{op} ,$$

where

$$Z_{op} = \exp[-\beta H + \alpha N] .$$

Here H is the Hamiltonian of the system, N is the particle number operator, $\beta^{-1} = kT$ (T being the temperature of the system and k the familiar Boltzmann constant) and α is related to the chemical potential μ by $\mu = \alpha/\beta$. The grand partition function may be considered as the fundamental function of the T -dependent MBP since, as is well known, all the thermodynamic properties of a many body system can be derived from Z .

The operator Z_{op} satisfies Bloch's equation:

$$\frac{\partial Z_{op}}{\partial \beta} = - H Z_{op} .$$

This equation is very similar in form to the Schrodinger equation. This similarity has been utilized by various authors to develop an expression for Z in terms of the mutual interaction potentials of the particles, e.g., Bloch and DeDominicis (1958) and Luttinger and Ward (1960) (these papers will be referred to as BDD and LW respectively). This procedure is mathematically similar to the method in field theory which

is known as the interaction picture and which was mentioned above in connection with the zero temperature case. In this approach Bloch and DeDominicis used an exact generalization of Wick contractions (Wick, 1950) applicable to the T -dependent system with the many body system considered in a grand ensemble. BDD's expression for the energy of the system bears strong resemblance to Goldstone's expression (Goldstone, 1957) for the ground state energy of the many fermion system. (In this dissertation we deal exclusively with fermions.) However, it was pointed out by Kohn and Luttinger (1960) that in actual fact the BDD expression does not reduce to that of Goldstone in the $T \rightarrow 0$ limit, Kohn and Luttinger showed that in taking the $T \rightarrow 0$ limit in the BDD expression, one gets, in addition to the Goldstone terms, the so-called anomalous terms.

Luttinger and Ward (1960) and Balian, Bloch and DeDominicis (1961) showed that for a central two body interaction (i.e. $V(x_i, x_j) = V(|x_i - x_j|)$ where $V(x_i, x_j)$ is the interaction potential between the particles i and j) the two expressions - Goldstone's and BDD's - can be brought into coincidence, i.e. the anomalous terms can be removed if one allows for the possibility that the chemical potential is different in the two cases.

In the LW presentation extensive use is made of the "single particle propagator." In their context this propagator traces what one might call "the propagation of the particle in temperature". In this formalism

a variable which is related to the temperature plays, formally, the role of time and the corresponding propagator is closely related to BDD contractions.

Matzubara (1955) was the first to show that it is possible to define a temperature dependent Green's function which possesses formally the same mathematical properties as the Green's function that was introduced in field theory by Schwinger (1951). A stimulating generalization of the Matzubara approach was worked out by Fradkin (1959) who considered grand ensemble averages. Fradkin showed that, in his formulation, the time-dependent Green's function for a temperature-dependent problem is most naturally defined as the analytic continuation of an appropriate T -dependent Green's function.

Apart from their aesthetic appeal, and the possibility that future developments may enhance their value, one might question whether the method of the temperature Green's functions is useful in its own right. That it is useful can be seen from the following argument. The expression for the single particle propagator in time (referred to as the "time Green's function") is known to have a direct physical meaning. This can be stated most succinctly in the following way: the poles of the time Green's function in the energy-momentum representation give the energy spectrum of the system. However, evaluation of the expressions obtained, when one attempts to solve the propagator-in-time equation directly, appears difficult if not impossible (Landau, 1958). Thus it is very useful to obtain via some other procedure expressions which seem

simpler and yet are capable of yielding the same information. For this reason the development in the temperature variable is of interest. It was shown by Abrikosov, Gorkov and Dzyaloshinskij (1959) (this paper will be referred to as AGD) that the retarded time Green's function or propagator is simply related to the temperature Green's function in the energy-momentum representation.

The present investigation was motivated by the striking similarity between the roles of the temperature and time variables in the many body problem. A central purpose of this work has been to investigate further this interesting similarity. In this connection we have investigated, in energy-momentum representation, "retarded and advanced" temperature Green's functions in analogy with the time Green's function along the lines given in Fradkin's paper (Fradkin, 1959). To this end a new formal scheme was devised whereby the parallelism of time and temperature is manifested in a clear way. For example, the formalism contains an "equation of motion" for the Green's function in the temperature variable. The analogy is further enhanced in Chapter III where it is shown amongst other things, that the temperature variable can formally be negative as well as positive. We were able in this way to obtain the fundamental equations of Luttinger and Ward in a new way, in particular LW's expression for the grand partition function is obtained in a concise way. The analogy between time and temperature Green's functions while close is not perfect. For example, in the energy-momentum representation (henceforth: Ep-rep) the analogy breaks down inasmuch as the retarded and advanced temperature Green's functions in this representation are

identical, whereas this is not true for the time Green's function. At this point it is shown (Chapter IV) that there is a natural way to split the temperature Green's function into two parts which are separately related to the Ep-reps of the advanced and retarded time Green's functions.

CHAPTER II: MATHEMATICAL PRELIMINARIES

(a) Introduction.

In this chapter we present a brief outline of that aspect of the formal theory of Green's functions which pertains to the expositions and methods of field theory. In section (b) the formalism of the functional differentiation is outlined; we will use this formalism to define the Green's functions appropriate to our problem. Section (c) contains a brief resume of Schwinger's variational principle, which is used to derive an explicit expression for the Green's function. In section (d) the salient results are summarized in a form that will be convenient in the succeeding chapters.

It might be appropriate at this point to make some descriptive remarks about the basic mathematical problem with which we are concerned in this chapter.

In the ordinary context of Green's functions one encounters linear differential operators and sources; the latter are completely specified at the outset. In the theory of interacting particles, however, one is faced with integro-differential equations (without external sources) which can be regarded as non-linear, homogeneous equations for the Heisenberg operators. Naturally, there are serious difficulties in extending the Green's function approach to this problem. The technique which will be elaborated in this chapter (Kato et al., 1960) is to

generalize the definition of the Green's function for linear operators to this non-linear case. In essence, the method is to introduce an auxiliary source as ^amatter of convenience - at the end of the calculation a limit is taken in which this source is set equal to zero. The generalized Green's functions thus obtained provide a convenient vehicle for expressing various quantities of physical interest.

(b) Functional Differentiation

The formalism of functional differentiation, which has been discussed by many authors (e.g. Schweber, 1961 pp. 193; Kato et. al., 1970; Akhiezer and Berestezki, 1953 pp. 160), provides an elegant formalism for the expression of formulae associated with the Green's function technique. This formalism provides a transparent "natural" generalization of the Green's function technique to the methods of field theory.

The functional derivative of the expression $\mathcal{F}[\phi(x)]$ with respect to $\phi(x')$ - denoted by $\delta\mathcal{F}[\phi]/\delta\phi$ - is defined by

$$\int \frac{\delta\mathcal{F}[\phi(x)]}{\delta\phi(x')} f(x') dx' \equiv \lim_{\epsilon \rightarrow 0} \frac{\mathcal{F}[\phi(x) + \epsilon f(x)] - \mathcal{F}[\phi(x)]}{\epsilon} \quad (1)$$

where $f(x)$ is reasonably arbitrary (e.g. a "smooth" and integrable function of x), and we can assume the domain of integration spans the region of definition of $\mathcal{F}[\phi(x)]$. The following pertinent formulae follow immediately from the definition (1):

$$\frac{\delta \phi(x)}{\delta \phi(x')} = \delta(x-x') \quad , \quad (2a)$$

$$\frac{\delta \int Q(x') \phi(x') dx'}{\delta \phi(x)} = Q(x) \quad . \quad (2b)$$

The Green's function associated with the equation

$$L \phi(x) = J(x) \quad , \quad (3a)$$

plays the role of the inverse of L in the sense

$$\phi(x) = \int G(x, x') J(x') dx' \quad . \quad (3b)$$

In equations (3) it has been assumed (and we shall continue to do so throughout) that L is non-singular (i. e. there are no non-trivial solutions to the equation $L\phi = 0$ consistent with the boundary conditions). Homogeneous boundary conditions at infinity are assumed throughout and the integration in (3b) is to be carried out over the entire 3-dimensional space. Equation (3b) serves as the definition of the Green's function associated with equation (3a) which satisfies the above-mentioned boundary conditions.

One can readily convince oneself that for a linear operator in (3a) the solution (3b) is such that the integral, in (3b), depends linearly on the source function $J(x)$ - or what amounts to the same thing, $G(x, x')$ is

is independent of $J(\mathbf{x})$ for linear operators. This observation along with formula (2b) leads to an alternative definition for the Green's function for a linear, non-singular operator:

$$G(x, x') = \frac{\delta \phi(x)}{\delta J(x')} \quad (4)$$

In order to generalize this formula to the case of non-linear operators we make use of the condition that the operators are non-singular; i. e. that $\phi \rightarrow 0$ as $J \rightarrow 0$. It can be shown that in the non-linear case the solution ϕ can be expanded in a "Taylor-like" series about $J = 0$, as follows (Kato et. al., 1960):

$$\phi(x) = \int \frac{\delta \phi(x)}{\delta J(x')} \bigg|_{J=0} J(x') dx' + \frac{1}{2} \int \frac{\delta^2 \phi(x)}{\delta J(x') \delta J(x'')} \bigg|_{J=0} J(x') J(x'') dx' dx'' + \dots \quad (5)$$

where we have used $\phi(x) \big|_{J=0} = 0$. That this, in fact, is a "natural" generalization of the familiar Taylor series can be seen, for example, by considering the first term in the above series in the case where usual partial differentiation is permissible. Thus, if we regard

$\phi(x)$, which is actually to be regarded as a functional of J , as a known function of J , the expression (5) is just the expansion of

$\phi(J)$ about the "point" $J=0$, and the first term corresponds to

$(\partial \phi / \partial J)_{J=0} J(x)$ as one would expect for a Taylor series. The

expansion (5) is useful if $J(\mathbf{x})$ is small enough to insure convergence. In the case where the source $J(\mathbf{x})$ is infinitesimally small it is expected that only the retention of the first term in (5) is necessary (Gottfried and Pičman, 1960). In this case we recover, for the non-singular and

non-linear operator, equation (4) in the following form:

$$G(x, x') = \left. \frac{\delta \phi(x)}{\delta J(x')} \right|_{J=0} . \quad (6)$$

We remark in summary that, for non-linear operators (e.g. for all non-trivial interaction operators in field theory) which are also non-singular, equation (6) provides a relatively simple formula for the Green's function for weak (or at least for infinitesimal) sources. One should note at this point that in the physical problem to be considered here the "source" should be regarded as an external potential.

The transcription of the Green's function formalism to field theory is most conveniently done when the latter is considered in the so-called Heisenberg picture, wherein the dynamics of the physical quantities are contained in the operators themselves. The physical reasons for this are discussed in Schweber (1961, pp. 651); for our purposes we merely remark that in field theory the analogue to the condition $\phi \rightarrow 0$ as $J \rightarrow 0$, as will be shown below (pp. 15, 16), is satisfied in the Heisenberg picture in a direct way. Now in the Heisenberg picture the equation of motion of the absorption operator $\psi(x)$, i.e. of the operator that removes a particle at the point $x(x = \underline{x}, t)$, is given by

$$(i\partial_t + \nabla_x^2)\psi(x) - \int \psi^\dagger(x')U(x,x')\psi(x')dx'\psi(x) \quad (7)$$

wherein $U(x, x')$ is the potential of the particle pair interaction and we have set $\hbar = 2m = 1$. This equation is a homogenous (non-linear) equation for $\psi(x)$ and should be compared with (3a) with $J(x) = 0$.^{*} It is convenient and customary (Kato et. al., 1960) to replace our interest in $\psi(x)$, in equation (7), by a consideration of $\langle \psi(x) \rangle$, i.e. of the expectation values of $\psi(x)$.

We remark here that this replacement has "physical sense" inasmuch as only the expectation values of an operator are meaningful physically. However, in the presence of a source we are still faced with the difficulty of defining $\langle \psi(x) \rangle$ uniquely, since the introduction of a source into the Lagrangian (Schwinger, 1951) has the result that the state vector of the system (the eigenvector of the Hamiltonian) is no longer stationary (we are interested in a nonstationary source - an external perturbing potential). This difficulty is overcome by considering the source to be time independent (for convenience we take it to be zero) at the beginning and at the end of the process under scrutiny. With these conditions on the source in mind, the expectation value of ψ can be defined in a more general way than indicated above.

* The bar on $\psi(x)$ is used as a reminder that we are dealing with Heisenberg field operators.

In fact, we will consider the following as the fundamental definition for the expectation value for any Heisenberg operator $Q(x)$:

$$\langle Q \rangle \equiv \text{Tr.} \{ \rho Q \} / \text{Tr.} \{ \rho \} , \quad (8a)$$

where

$$\rho \equiv \sum_m |m_f\rangle \exp[-\beta E_m^N + \alpha N] \langle m_i| , \quad (8b)$$

m_i being an initial, and m_f a final state vector of the system in the Heisenberg picture. A detailed discussion of the meaning of the summation is given in Appendix A. In the above definition (equations (8)) we have specialized to a system in a grand canonical ensemble which is the only one considered in this dissertation. In equations (8), E_m^N is the energy eigenvalue of the m -th state with N particles present, $\beta^{-1} = kT$, k being Boltzmann's constant, and α is related to the chemical potential μ through $\mu = \alpha/\beta$. Note that if the spectrum of initial and final states is identical, then ρ is the usual trace of the density operator for a grand canonical ensemble (see, e.g., ter-Haar, 1955, pp. 149).

We now focus our attention on (spin indices are suppressed throughout)

$$- \left. \frac{\delta \langle \psi(x) \rangle}{\delta J(x')} \right|_{J=0} \equiv G(x, x') \quad (9)$$

This equation defines the Green's function of interest to us in this investigation. We note immediately that $\langle \psi \rangle \rightarrow 0$ as $J \rightarrow 0$, i.e. the non-singular condition is satisfied because of the properties of the absorption operator $\psi(x)$ and the orthogonality of the Heisenberg state vectors. It is of interest to mention that the discussion presented here (Kato et. al., 1960) is also of current theoretical interest in problems involving the so-called "response" of a system to infinitesimal disturbances, e.g. in the study of the stability of physical systems (Gottfried and Pičman, 1960).

In order to get a manageable expression for the left-hand side of equation (9), appeal must be made to the physical laws governing the variation of Heisenberg operators. A convenient variational principle for this purpose is Schwinger's dynamical principle which is outlined in the next section.

(c) Schwinger's Dynamical Principle.

The dynamical principle of Schwinger (1951, 1953) is summarized by the equation

$$\delta \langle \alpha | \beta \rangle = i \langle \alpha | \delta \int \mathcal{L} d^4x | \beta \rangle \quad (10)$$

where \mathcal{L} is the Lagrangian density of the problem, $|\alpha\rangle$ is the final and $|\beta\rangle$ the initial state vector of the system. In words, equation (10) states that a small variation of the external sources (i.e. of appropriate terms in \mathcal{L}) results in a variation of the transition probability amplitude $\langle \alpha | \beta \rangle$.

The prescription (10) was shown by Schwinger (1951, 1953) to be the generic formula for all the fundamental equations and relations in quantum mechanics. In particular, a simple demonstration of the equivalence of this prescription to the Feynman formulation of quantum-mechanics (Feynman, 1948) is outlined in the review paper of Kato et. al. (1960).

Note that the transition amplitudes $\langle \alpha | \beta \rangle$ in equation (10) refer to pure states and are distinct from the canonical averages of Heisenberg operators which occurred in equation (8). Here $|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of the Hamiltonian. In the present investigation we are concerned with a system which is characterized by a given temperature and whose particle number can vary. Thus we cannot characterize the system by an eigenvector of the Hamiltonian

either before the introduction of the source or after the source is switched off. In fact this was the reason that we used statistical expectation values as defined by equations (8).

Naturally, a 'source term' in an equation for ψ must be considered as deriveable from a change in the Lagrangian density of the Fermi field. Thus, in addition to the Lagrangian appropriate to the actual system, we include a term of the form

$$- \int^+ (x) \psi(x) - \psi^+(x) J(x)$$

where $J(x)$ is an arbitrary external perturbation which anticommutes with $\psi(x)$. With this change in the Lagrangian density we get from equation (10), the following equation (the proof and references are given in Appendix A):

$$G(x, x') \equiv - \left. \frac{\delta \langle \psi(x) \rangle}{\delta J^+(x')} \right|_{J=0} = i \langle T \psi(x) \psi^+(x') \rangle \quad (11)$$

where the angular brackets are to be interpreted by equations (8), and where T is Wick's chronological operator (Wick, 1950):

$$\begin{aligned} T \psi(x) \psi^+(x') &= \psi(x) \psi^+(x') & \text{for } t > t' \\ &= - \psi^+(x') \psi(x) & \text{for } t < t', \end{aligned}$$

In the next section the results to this point are summarized and an expression for the Gibbs potential in terms of the Green's function is written down.

(d) The Equation of Motion for the Green's Function.

We start with a brief summary of the results obtained so far. The usual Green's function for linear, non-singular differential operators which act on a scalar function $\phi(x)$ (i.e. $\phi(x)$ is a one component function which is scalar with respect to rotations in ordinary 3-dimensional space) in the presence of a source $J(x)$, was given (Kato et. al., 1960) in terms of functional differentiation:

$$G(x, x') = \frac{\delta \phi(x)}{\delta J(x')} \quad (\text{Linear case}). \quad (4)$$

The case of non-linear, non-singular operators operating on a scalar function with infinitesimal scalar sources was shown to lead to an analogous equation:

$$G(x, x') = \left. \frac{\delta \phi(x)}{\delta J(x')} \right|_{J=0} \quad (\text{Non-linear case}). \quad (6)$$

In the derivation of formula (6) it was assumed that the solution ,

$\phi(x)$, depends linearly on the source (we have simply neglected the higher order terms -- which is presumably justifiable for infinitesimal sources). At this point, rather than considering the Heisenberg operator $\psi(x)$, we consider the equation for $\langle \psi \rangle$, where the expectation value was defined to be the grand canonical average (equation 8). (It should be mentioned here that a formulation

based on petit ensemble average has been published (Brout and Englert, 1960) -- however, the petit ensemble approach to the problem is much more complicated than that of the grand ensemble. The Green's function was defined to be

$$-G(x, x') \equiv \left. \frac{\delta \langle \psi(x) \rangle}{\delta J(x')} \right|_{J=0}, \quad (9)$$

which is a "natural" generalization of equation (6), Schwinger's dynamical principle was then utilized to obtain an explicit formula for $G(x, x')$ in terms of the Heisenberg absorption and creation operators $\psi(x)$ and $\psi^\dagger(x)$:-

$$G(x, x') = i \langle T[\psi(x) \psi^\dagger(x')] \rangle. \quad (11)$$

To obtain the equation of motion satisfied by the Green's function we now proceed as follows. Consider the so-called Schwinger identity (Klein and Prange, 1958):

$$\partial_t T[\psi(x) \psi^\dagger(x')] = T[\partial_t \psi(x) \psi^\dagger(x')] + \delta^4(x-x'). \quad (12)$$

(This equation is proven in a simple way in Appendix A.) Equations (12) and (11) coupled with the equations of motion of the Heisenberg operator,

$$i \partial_t \psi(x) = [\psi(x), H]_-, \quad (13)$$

lead to the equation of motion for the Green's function (H is given by equation 21):

$$\begin{aligned} (i\partial_t + \nabla_r^2) G(x, x') - i \int d^3 x'' U(x'', x) \langle T [\psi(x'') \psi(x) \psi^\dagger(x'') \psi^\dagger(x')] \rangle \\ = -\delta^4(x - x'). \end{aligned} \quad (14)$$

The expression in angular brackets can be simplified in the following manner. We define a dimensionless parameter λ (e.g. see Klein and Prange, 1958) which characterizes the interaction, $U(x, x') \rightarrow \lambda U(x, x')$. Furthermore we define the "irreducible self energy operator" (Klein and Prange, 1958; Klein, 1960; Klein, 1961) by the equation

$$-\int d^4 x'' M(x, x'') G(x'', x') \equiv i\lambda \int d^3 x'' U(x'', x) \langle T [\psi(x'') \psi(x) \psi^\dagger(x'') \psi^\dagger(x')] \rangle. \quad (15)$$

Since our discussion will eventually lead to a perturbation expansion, it is useful to define the "non-interacting" Green's function, $G^{(0)}(x, x')$. This function is defined by the equation:

$$(i\partial_t + \nabla_r^2) G^{(0)}(x, x') = -\delta^4(x - x'), \quad (16)$$

i.e., the equation obtained by setting $U = 0$ in (14). We now insert (15) into (14) and rewrite the resultant equation in the form

$$(i\partial_t + \nabla_r^2) G(x, x') + \int d^4 x'' d^4 x''' \delta^4(x'' - x) M(x''', x'') G(x'', x') = -\delta^4(x - x'). \quad (17)$$

The definition (16) can now be used to convert (17) into an integral equation. The result is (Klein and Prange, 1958)

$$G(x, x') = G^{(0)}(x, x') + \int d^4x_1 d^4x_2 G^{(0)}(x, x_1) M(x_1, x_2) G(x_2, x'). \quad (18a)$$

This equation can be written in the more suggestive form

$$G = G^{(0)} + G^{(0)} M G, \quad (18b)$$

$$= G^{(0)} + G^{(0)} M G^{(0)} + \dots \quad (18c)$$

The last expression, (18c), is seen to be a perturbation expansion; in fact, as we shall see in Chapter III, it bears a close similarity to equation (38) in the Luttinger and Ward paper.

We pause here for a general qualitative remark on the importance of the assumption that our problem can be solved by perturbation procedures (i.e., by terminating the expansion (18c)). The physical significance of this assumption can be seen in that it rules automatically against the possibility of a bound fermion pair of the Cooper type (Cooper, 1956). This is because the energy gap term (Δ in Cooper's notation) can be obtained from the methods of field theory only by summing an infinite number of terms in an equation like (18c). Thus, the summation indicated by (18c) can be written formally as:

$$G = \frac{1}{[G^{(0)}]^{-1} - M},$$

wherein, crudely speaking, M plays the role of Δ in Cooper's theory and $[G^{(0)}]^{-1}$ is the operator inverse to the non-interacting Green's function. At the same time it is well-known that the accepted theory for superconductivity, the so-called BCS theory (Bardeen et. al., 1957) rests on the existence of "Cooper states". Thus, in this case, the usual perturbation procedure is unable to unravel the physical situation. In general it is to be expected that the energy discontinuities and other processes involving phase changes (such as was suggested by Trainor and Dixon for nuclear matter, (1955)) cannot be explained by a perturbation expansion. The fact that perturbation techniques are, nonetheless, widely used in many body theory expresses in part the difficulty of those problems from both mathematical and conceptual points of view; on the other hand, it is relatively easy to assign a pictorial meaning to the perturbation terms involved. Apart from this, perturbation expansions are useful because the very nature of their failure in a given problem helps to delineate those features which are characteristically important for that problem. As an example consider once again the theory of superconductivity: if one attempts a perturbation expansion for the electron interactions in metals, the second order terms have a singularity at the fermi surface. The failure of perturbation techniques led eventually to Cooper's theory which provided a sound basis for the description of the phenomenon of superconductivity.

The expression for the energy of a system in a grand canonical ensemble is given by (cf., e.g., ter Haar, 1955 pp. 149)

$$E = \frac{\text{Tr.} \{ H \exp[-\beta(H - \mu N)] \}}{\text{Tr.} \{ \exp[-\beta(H - \mu N)] \}} \equiv \langle H \rangle, \quad (19)$$

where the last identity expresses our general definition for expectation values (equation 8). (Note that this quantity is evaluated with an infinitesimal source J and then the source is set equal to zero.) The chemical potential, μ , is defined as usual by the requirement that

$$[\text{The average number of particles}] = \langle N \rangle. \quad (20)$$

The Hamiltonian, H , in the Heisenberg picture can be written as ($J = 0$)

$$H = \int d^3x : \nabla \psi^\dagger(x) \nabla \psi(x) : + \frac{1}{2} \int d^3x d^3x' : \psi^\dagger(x') \psi^\dagger(x) U(x', x) \psi(x) \psi(x') : , \quad (21)$$

wherein terms enclosed by colons are to be taken in the "normal" order (Wick, 1950); i.e., in any sequence of operators, all referring to the same time, absorption operators are to be brought to the right of creation operators. (We assume throughout that the interaction is instantaneous: i.e., two particles interact only at the same time.)

We shall not discuss the reason for having the operators in their normal order (Wick, 1950), but merely remark that in this way some unobservable divergences are automatically removed from the theory at the outset (Dyson, 1952, pp. 156).

In closing this review chapter, we would like to outline the derivation of an alternative to the usual expression for the Gibbs potential. The Gibbs potential, A , is usually defined by

$$\text{Tr.} \{ \exp[-\beta(H - \mu N)] \} = \exp[\beta A] , \quad (22a)$$

or

$$A = \frac{1}{\beta} \ln. \{ \text{Tr.} (\exp[-\beta(H - \mu N)]) \} . \quad (22b)$$

The alternative expression for A which is of interest to us is obtained as follows: we characterize the interaction by the replacement $\stackrel{(p.21)}{\underset{\wedge}{U}}(\mathbf{x}, \mathbf{x}) \rightarrow \lambda U(\mathbf{x}, \mathbf{x})$ and thus write

$$H = H_0 + \lambda H_1 ,$$

$$\lambda H_1 = \frac{\lambda}{2} \int d^3x' d^3x : \psi^\dagger(x) \psi^\dagger(x') U(x, x') \psi(x') \psi(x) : ,$$

and upon differentiation with respect to λ , we obtain (cf. Klein, 1961a)

$$\left. \frac{\partial A}{\partial \lambda} \right|_{\beta, V, \mu} = \langle H_1 \rangle \quad (23)$$

Converting (23) into an integral equation, it reads

$$A = A \Big|_{\lambda=0} + \int_0^\lambda \langle H_1 \rangle d\lambda' . \quad (24)$$

This equation leads, via the definition equation (15), to our alternative expression for the Gibbs potential:

$$A = A \Big|_{\lambda=0} - \frac{1}{2} i \int_0^\lambda \frac{d\lambda'}{\lambda'} \int d^3r d^4x' M(x, x') G(x', x) . \quad (25)$$

This equation was obtained by Klein (1961b).

It is important to note that $A \Big|_{\lambda=0}$ involves the chemical potential as determined by equation (20). That μ , the chemical potential, might be different from the chemical potential of free fermions (labelled by μ_0) forms a cornerstone in the LW scheme for elimination of the so-called anomalous diagrams.

In the next chapter it will be shown that equation (25) leads to the basic equation (45) in LW's paper for the Gibbs potential, although (25) was derived by considering temporal developments while LW used the BDD approach (cf. p. 3).

CHAPTER III: TIME AND TEMPERATURE GREEN'S FUNCTIONS

Introduction.

In the previous chapter we outlined the formal derivation of the expression for the single particle Green's function. We shall refer to this expression (equation (9)) as the "time" Green's function. It is well known that (c f., e.g., Thouless, 1961) the "time" Green's function lends itself to a relatively simple interpretation if free particles are considered and complications due to statistical averaging are deleted. In this case one can consider the non-interacting "time" Green's function as the scalar product of the ground state with the state which is formed from the ground state by the addition of a particle at the time t , allowing the particle to propagate freely until t' , and then removing a particle from the system. (In the above interpretation we have assumed that the creation operator was to the right of the absorption operator; it is clear, however, that a similar interpretation can be presented for the reverse situation.) It is possible to show that (c f., e.g., Thouless loc. cit.) the expectation value of any single particle operator can be obtained directly from the "time" Green's function. We shall not attempt the interpretation for the interacting Green's function in a statistical ensemble; however, this can be done by "stretching" somewhat the above discussion.

We are unable to give an interpretation for the "temperature" Green's function which is alluded to in the title of this chapter. Thus it must be considered as a formal device wherein one encounters what we will refer to as a "temperature variable." This variable, which is customarily denoted by u , has the dimension of reciprocal temperature (i.e. reciprocal energy) and, in our formalism, is bounded by $-\beta/2 \leq u \leq +\beta/2$ where $\beta^{-1} = kT$.

The parallelism between the roles of the temperature variable and the time in the Green's function formulation of the temperature dependent MBP was first discussed by Matzubara (1955) who considered systems with a fixed number of particles. Matzubara's approach was extended by Fradkin (1959) and AGD (1959) to a system in a grand ensemble.

On the other hand it was the perturbation-like approach of BDD (1959) which was used by LW (1960) for their discussion of the correct generalization of the Goldstone (1957) expansion for systems at finite temperatures.

Klein (1960, 1961a) obtained the generalization of the Goldstone expansion for systems at finite temperatures by using the "time" Green's function approach. Furthermore Klein (1961b) was able to derive Landau's semiphenomenological theory of Fermi liquids

(Landau, 1957) by an extensive use of the "time" Green's function.

In this chapter the temperature dependent MBP is reformulated in terms of a new variable τ which will be shown to play the role of either the time (t) or the temperature variable (u); this new variable replaces the time in the theory outlined in the previous Chapter. In this way the intimate interchangeability between t and u is shown explicitly. Further, our formulation permits the assignment to u of negative as well as positive values and thus enhances its similarity to the time. We prove the validity of our formulation by deriving, using the "temperature" Green's function, the fundamental equations of LW for the Gibbs potential and the single particle propagator.

(b) Mathematical Preliminaries.

Our approach involves the following substitution:

$$t \rightarrow \tau \equiv \begin{pmatrix} t & 0 \\ 0 & iu \end{pmatrix}, \quad (1)$$

where t is interpreted as the usual time variable while u is to be regarded as a "temperature variable" with the dimension of reciprocal temperature, i.e. reciprocal energy. This substitution has the following consequences: -

(i)

$$\psi(\underline{r}, t) \rightarrow \psi(\underline{r}, \tau) = \begin{pmatrix} \psi(\underline{r}, t) & 0 \\ 0 & \psi(\underline{r}, iu) \end{pmatrix}, \quad (2)$$

where $\psi(\underline{r}, t)$ is ^{an} arbitrary, reasonably well behaved function, for example, any function which has a Fourier expansion:

$$\psi(\underline{r}, \tau) = \sum_p a_p(\underline{r}) e^{ip\tau} = \sum_p a_p(\underline{r}) \left[I + ip\tau + \frac{(ip\tau)^2}{2!} + \dots \right]$$

$$= \sum_p a_p(\underline{r}) \begin{pmatrix} e^{ipt} & 0 \\ 0 & e^{ipu} \end{pmatrix} = \begin{pmatrix} \psi(\underline{r}, t) & 0 \\ 0 & \psi(\underline{r}, iu) \end{pmatrix}.$$

(ii) If τ is subjected to a similarity transformation through a 2×2 matrix,

$$\tau \rightarrow \hat{\tau} = S \tau S^{-1},$$

then it follows immediately from formula (2) that

$$\psi(r, \hat{t}) = S \psi(r, \tau) S^{-1} . \quad (3)$$

(iii)

$$\partial_t \rightarrow \partial_\tau = \begin{pmatrix} \partial_t & 0 \\ 0 & -i \partial_u \end{pmatrix} . \quad (4)$$

This formula follows in an obvious way from equation (2). Its consistency can be seen, for example, in

$$\partial_\tau \tau = I .$$

(iv) We now regard τ as a variable conjugate to the Hamiltonian; hence for Heisenberg operators such as $\psi(r, \tau)$ we have

$$i \partial_\tau \psi = [\psi, H]_- , \quad (5)$$

where, in addition to normal commutation, T - multiplication of ψ and H is implied.

(v) The chronological operator T operating on $\psi(r, \tau) \psi^\dagger(r', \tau')$ is now interpreted as

$$T[\psi(r, \tau) \psi^\dagger(r', \tau')] = \begin{pmatrix} T_\tau[\psi(r, \tau) \psi^\dagger(r', \tau')] & 0 \\ 0 & T_u[\psi(r, iu) \psi^\dagger(r', iu')] \end{pmatrix} \quad (6)$$

where in each matrix element the operation of T is the usual one (Wick, 1950); thus for the field operators ψ and ψ^\dagger :

$$\begin{aligned} T_\tau[\psi(r, \tau) \psi^\dagger(r', \tau')] &= \psi(r, \tau) \psi^\dagger(r', \tau') \quad t > t' \\ &= -\psi^\dagger(r', \tau') \psi(r, \tau) \quad t < t', \end{aligned}$$

and

$$\begin{aligned} T_u[\psi(r, iu) \psi^\dagger(r', iu')] &= \psi(r, iu) \psi^\dagger(r', iu') \quad u > u' \\ &= -\psi^\dagger(r', iu') \psi(r, iu) \quad u < u'. \end{aligned}$$

(c) The Equation of Motion of the Green's Function.

(i) The Schwinger Identity:

The Schwinger identity can be obtained in the usual way.

From the definition of T, we have

$$T[\psi(r, \tau) \psi^\dagger(r', \tau')] \equiv \theta(\tau - \tau') \psi(r, \tau) \psi^\dagger(r', \tau') - \theta(\tau' - \tau) \psi^\dagger(r', \tau') \psi(r, \tau).$$

Here

$$\theta(\tau) = \begin{pmatrix} \theta(t) & 0 \\ 0 & \theta(iu) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \theta(t) &= \begin{cases} 1 & t > 0 \\ 0 & t < 0, \end{cases} \\ \theta(iu) &= \begin{cases} 1 & u > 0 \\ 0 & u < 0. \end{cases} \end{aligned}$$

Hence (cf. Appendix A)

$$\partial_\tau T[\psi(r, \tau) \psi^\dagger(r', \tau')] = \delta(\tau - \tau') \delta(r - r') + T[\partial_\tau \psi(r, \tau) \psi^\dagger(r', \tau')],$$

where we have used the anti-commutation relation for the Heisenberg operator ψ and ψ^\dagger .

(ii) The Equation of Motion of the Heisenberg Operator ψ .

Using equation (5) we get

$$i \partial_\tau \psi = - \nabla^2 \psi + \lambda \int d^3 r' U(|\vec{r} - \vec{r}'|) \psi(\vec{r}', \tau) \psi(\vec{r}', \tau) \psi(\vec{r}, \tau), \quad (7)$$

where H is given by equation (21) of Chapter II. We note that the operators involved in (7) all refer to the same τ . The meaning of the colons is the same as in Chapter II (equation 21); however, here all the formulae are matrix relationships.

(iii) Definition of Expectation Values in a Grand Canonical Ensemble.

In the formalism used here the Heisenberg state vectors must be considered as having two components. Thus, our (2x2) matrix operator has two eigenvectors corresponding to each eigenvector of the usual Hamiltonian. We shall use the notation

$$\uparrow \equiv \begin{pmatrix} m \\ 0 \end{pmatrix} \quad \text{and} \quad \downarrow \equiv \begin{pmatrix} 0 \\ m \end{pmatrix}.$$

Here m denotes the Heisenberg state vector: $H|m\rangle = E|m\rangle$ and $N_0|m\rangle = N_m|m\rangle$ where H and N_0 are the usual Heisenberg Hamiltonian and particle number operators. Either one of the above states (i.e., \uparrow or \downarrow) can be used to obtain all the properties of the system as will become clear below. However, to retain the generality of our approach we will use the notation

$$m \equiv \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

however all results should be interpreted as either \uparrow or \downarrow . In this way the formulae can be developed in a unified fashion. With this preamble we can proceed with the pertinent definitions. The expectation value of an arbitrary operator Q is defined by its grand canonical average: -

$$\langle Q \rangle \equiv \text{Tr} \{ \rho Q \} / \text{Tr} \{ \rho \}, \quad (8a)$$

with

$$\rho \equiv \sum_m |m\rangle \omega_m \langle m|, \quad (8b)$$

and

$$\omega_m \equiv \exp[-\beta E + \alpha N], \quad (8c)$$

where the terms are defined in the same manner as in conjunction with equation (8) of Chapter II.

(iv) Equation of Motion of the Single Particle Green's Function:

The single particle Green's function can now be defined in the usual way: -

$$G(x, x') = i \langle T [\psi(x) \psi^\dagger(x')] \rangle \quad (9)$$

where $x \equiv (\underline{r}, \tau)$.

Using equations (6) (7) (8) (9) we get(cf. the derivation of equation (14) in Chapter II) the equation of motion of the Green's function:

$$(-i\partial_\tau - \nabla^2) G(x, x') = \delta(x - x') - i K(x, x') \quad (10)$$

where

$$K(x, x') = \lambda \left\langle T \int d^3 \underline{r}'' U(\underline{r}'' - \underline{r}) \psi^\dagger(\underline{r}'', \tau) \psi(\underline{r}'', \tau) \psi(\underline{r}, \tau) \psi^\dagger(\underline{r}', \tau') \right\rangle, \quad (11)$$

which is, of course, formally equivalent to equation (14) of Klein and Prange and equation (14) of Chapter II. The equation of motion of the non-interacting single particle Green's function, $G^{(0)}(x, x')$, is given by

$$(-i\partial_\tau - \nabla^2) G^{(0)}(x, x') = \delta(x - x') \equiv \delta(\tau - \tau') \delta(\underline{r} - \underline{r}'), \quad (12)$$

i. e. by setting $\lambda = 0$ in equation (10).

(v) The Iterative Equation for the Green's Function.

As in Chapter II (section (d)) we define the "irreducible self energy operator" by the equation:

$$i\lambda \left\langle T \int d^3\vec{r} U(\vec{r}-\vec{r}') \psi^\dagger(\vec{r},\tau) \psi(\vec{r},\tau) \psi(\vec{r},\tau) \psi^\dagger(\vec{r}',\tau) \right\rangle \quad (13)$$

$$\equiv \int d\vec{r} d^3\vec{r}'' M(x,\vec{x}) G(\vec{x},\vec{x}') \quad .$$

Substituting (13) into (10), the latter can be converted into an integral equation upon utilization of (12). The result is

$$G(x,x') = G^{(0)}(x,x') + \int d^4\vec{x} d^4\vec{x}'' G^{(0)}(x,\vec{x}) M(\vec{x},\vec{x}') G(\vec{x},\vec{x}') \quad , \quad (14a)$$

$(d^4x \equiv d\tau d^3\vec{r})$. As before we can write equation (14a) in the more suggestive form

$$G = G^{(0)} + G^{(0)} M G \quad . \quad (14b)$$

Equation (14b) is the iterative equation for the single particle Green's function.

In the next section we will prove the validity of the equations given in this section.

(d) Proof of the Validity of the Present Formulation.

If one regards the expectation values that were considered in

the previous section as taken with respect to the states designated by \uparrow (i.e. $\begin{pmatrix} 11 \\ 0 \end{pmatrix}$) then, clearly, all the equations considered in that section become identical with those considered in the previous chapter. Hence, this case requires no further discussion. The validity of our equations does require special attention when we consider the expectation values as taken with respect to the states designated by \downarrow (i.e. $\begin{pmatrix} 0 \\ 11 \end{pmatrix}$) of the previous section. The verification of the expressions for this case will be provided below by utilizing our equations to derive the fundamental equations of LW for the Gibbs potential. Having established this equivalence, one can then turn to the analysis of Luttinger and Ward for a physical interpretation of the mass operator.

(i) The Temperature Green's Function for a Free Particle in the Energy-Momentum Representation.

The free particle Green's function is obtained by setting $\lambda = 0$ in the expression for the interacting Green's function, i.e. equation (12) Chapter II. The result is

$$G^{(10)}(x, x') = \frac{-i \text{Tr} \left\{ T \left[\psi(x) \psi^\dagger(x') \right] \exp \left[-\beta (H_0 - 4N) \right] \right\}}{\text{Tr} \left\{ \exp \left[-\beta (H_0 - 4N) \right] \right\}} \quad (15)$$

It might be emphasized that in the present context $x = (r, iu)$. We now expand the operators $\psi(x)$ and $\psi^\dagger(x)$ by Fourier series - it is implicitly assumed throughout that this is possible

(cf. equation (2), p. 30). We limit ourselves to consideration of Green's functions which depend upon the spatial differences $(\underline{r} - \underline{r}')$ only; this is equivalent to the assumption of a uniform medium (the significance of this assumption is discussed below). We then obtain

$$G^{(0)}(x, x') = \frac{-i}{V} \sum_p \frac{\exp[i p \cdot (\underline{r} - \underline{r}') + \varepsilon_p(u - u')]}{1 + \exp[\beta(\varepsilon_p - u)]} \quad \text{for } u > u', \quad (16a)$$

$$G^{(0)}(x, x') = \frac{i}{V} \sum_p \frac{\exp[i p \cdot (\underline{r} - \underline{r}') + \varepsilon_p(u - u')]}{1 + \exp[-\beta(\varepsilon_p - u)]} \quad \text{for } u < u', \quad (16b)$$

where V is the volume of the system and periodic boundary conditions have been assumed. The details of the calculation leading to equations (16) will be given in Chapter IV.

We would like to pause at this juncture to discuss the assumption of uniform density that was used in obtaining equations (16). The possibility that self-interacting systems of identical particles may possess a non-uniform density has been considered in the literature. For example, Trainor, Bhatia and Sood (1958) considered a periodic model for nuclear matter and were able to calculate values of the

nuclear density and the binding energy per nucleon in good agreement with experimental values. The possibility of obtaining density fluctuations in many body fermion systems has been recently investigated by several authors (Overhauser (1960), Kohn and Nettel (1960), Gross (1960), Brout (1960), Henley and Ruijrok (1960), Werner (1961), Thouless (1961) who were confined, because of computational difficulties, to the limit of weak attractive interactions. However, attention has been restricted, for the most part, to uniform media; this is essentially because of the appreciable difficulties involved in describing the development of density fluctuations from uniformity as the interaction between particles is turned on.

(ii) Derivation of the LW Representation for $G^{(0)}(x, x')$.

We are now in a position to obtain the propagator of LW from our equations (16) using, in fact, a technique devised by LW themselves. To accomplish this, we need to restrict the variable u ; we impose on u the condition $|u| \leq \beta/2$ where β is the usual inverse temperature factor. Note that u can take on negative as well as positive values. We note that in equations (16), the factors in the terms which contain ξ_p explicitly are analytic functions of ξ_p . It is expedient now to analytically continue these factors into the complex ξ_p plane, which we label by \mathcal{F} . We can now represent the terms which contain ξ_p explicitly in equations (16) as follows:

$$\frac{\exp[+\varepsilon_p(u-\dot{u})]}{1 + \exp[\beta(\varepsilon_p - \mu)]} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\exp[\zeta(u-\dot{u})]}{1 + \exp[\beta(\zeta - \mu)]} \cdot \frac{d\zeta}{\zeta - \varepsilon_p} \quad \text{for } u > \dot{u}, \quad (17a)$$

$$\frac{\exp[-\varepsilon_p(\dot{u}-u)]}{1 + \exp[-\beta(\varepsilon_p - \mu)]} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\exp[-\zeta(\dot{u}-u)]}{1 + \exp[-\beta(\zeta - \mu)]} \cdot \frac{d\zeta}{\zeta - \varepsilon_p} \quad \text{for } u < \dot{u}, \quad (17b)$$

where Γ_0 is defined in Fig. 1:

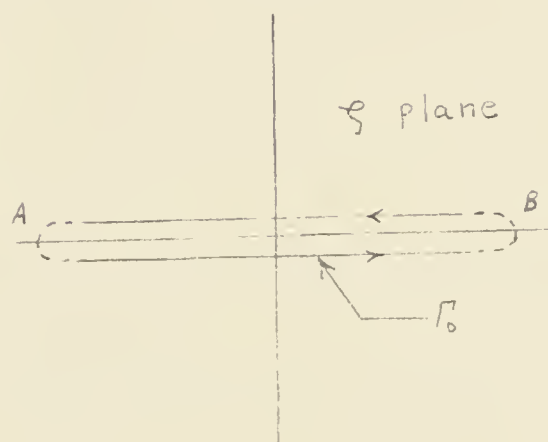


Fig. 1: The Contour Defining Γ_0 .

In Fig. 1 the path is chosen to be below (above) the poles of the integrands in the upper (lower) half of the complex ζ plane. With this choice of Γ_0 the proof of these integral representations follows trivially from Cauchy's theorem, there being no poles on the

real axis except at $\zeta = \xi_f$. Now the integrands in equations (17) are zero at A and B in Fig. 1 (i.e. at $\pm \infty$); further, if we analytically continue the integrands, as functions of ζ , into the whole complex plane we note that they are zero on the large semicircles above and below the real axis. Hence we can replace the contour Γ_0 by the contours Γ which are shown in Fig. 2 (This technique is due to LW).



Fig. 2: The Contours Defining Γ .

In other words we have replaced the residues of the integrands in equation (27) on the real axis by their residues in the whole complex plane - except for the real axis. Here the advantage of this procedure becomes apparent - the equations (16) take on an identical form:

$$G^{(0)}(x, x') = \frac{i}{\beta V} \sum_p \sum_\ell \frac{\exp[i p \cdot (\underline{r} - \underline{r}') - \xi_\ell(u - u')]}{\xi_\ell - \varepsilon_p} , \quad (18)$$

$$\xi_\ell = \frac{2\ell+1}{\beta} \pi i + \mu ; \ell = 0, \pm 1, \pm 2, \dots .$$

With this expression we can label the single particle non-interacting Green's function in the energy-momentum representation by

$$G^{(0)}(\xi_\ell, \varepsilon_p) = \frac{1}{\xi_\ell - \varepsilon_p} , \quad (19)$$

i. e. we consider the set $\{ \phi_{p,\ell}(\underline{r} - \underline{r}'; u - u') \}$, where

$$\phi_{p,\ell}(\underline{r}, v) \equiv \exp[ip \cdot \underline{r} - \xi_\ell v] , \quad (20)$$

to be a complete set in the sense that $G(x, x')$, $G^{(0)}(x, x')$ and $M(x, x')$ can be expanded in terms of it in the 4 dimensional range: $2\beta V$.

This point is discussed further in Appendix B.

(iii) Derivation of LW's Fundamental Equations.

The formula connecting the non-interacting Green's function with the interacting Green's function and the "irreducible self energy operator" was obtained in section (d) (v) of this Chapter:

$$G(x, x') = G^{(0)}(x, x') + \int d^4x_1 d^4x_2 G^{(0)}(x, x_1) M(x_1, x_2) G(x_2, x') . \quad (14a)$$

Expanding the operators in this equation in the set introduced above (equation 20) leads to (after the space integrations, which are done trivially):

$$\begin{aligned} \frac{i}{\beta V} \sum_P \sum_\lambda G(\beta_\lambda, \varepsilon_P) \phi_{P,\lambda}(\underline{r}-\underline{r}'; u-u') = \\ \frac{i}{\beta V} \sum_P \sum_\lambda G^{(0)}(\beta_\lambda, \varepsilon_P) \phi_{P,\lambda}(\underline{r}-\underline{r}'; u-u') \\ - \frac{i}{\beta^3 V} \sum_P \sum_{\lambda, m, \lambda'} G^{(0)}(\beta_\lambda, \varepsilon_P) M(\beta_m, \varepsilon_P) G(\beta_{\lambda'}, \varepsilon_P) \exp[-\beta_\lambda u + \beta_{\lambda'} u'] \times \\ \times \int idu_1 idu_2 \exp[u_1(\beta_\lambda - \beta_m)] \exp[u_2(\beta_m - \beta_{\lambda'})] . \end{aligned} \quad (21)$$

Now the integrations of the variables u_1 and u_2 over the range $-\beta/2$ to $+\beta/2$ can be readily carried out with the result:

$$-\beta^2 \delta_{\xi_\ell, \xi_m} \delta_{\xi_m, \xi_{\ell'}} .$$

With this, equation (21) takes the following form:

$$\frac{i}{\beta V} \sum_{p, \ell} \phi_{p, \ell}(\underline{r} - \underline{r}'; u - u) \left[G(\xi_\ell, \xi_p) - G^{(0)}(\xi_\ell, \xi_p) - G^{(0)}(\xi_p, \xi_p) M(\xi_p, \xi_p) G(\xi_p, \xi_p) \right] = 0 ;$$

and, because of the linear independence of the $\phi_{p, \ell}$, we get (deleting the indices ξ_p and ξ_ℓ)

$$G = G^{(0)} + G^{(0)} M G , \quad (22)$$

which is identical in form with equation (14b), except that here the terms are in the energy-momentum representation. Utilizing equation (19) we can rewrite equation (22) in form

$$G(\xi_\ell, \varepsilon_p) = \frac{1}{\xi_\ell - \varepsilon_p - M(\xi_\ell, \varepsilon_p)} . \quad (23)$$

Equations (22) and (23) should be compared with equations (38) and (39) in the paper of LW.

We are now in a position to derive the LW equation for the Gibbs potential (equation (45) in LW's paper). Recalling that we had (Chapter II equation (25)) the following expression for the Gibbs potential A ,

$$A = A|_{\lambda=0} + \frac{i}{2} \int_0^\lambda \frac{d\lambda'}{\lambda'} \int d^3r d^4x' M(x, x') G(x', x) . \quad (25)$$

We now expand once more the functionals M and G in terms of $\phi_{p,\ell}$ in equation (20) and get, after straight-forward integration

$$A = A|_{\lambda=0} + \frac{1}{2\beta} \sum_p \sum_\ell \int \frac{d\lambda'}{\lambda'} M(\xi_\ell, \varepsilon_p) G(\xi_\ell, \varepsilon_p) . \quad (24)$$

With the aid of (19), (23), and (24) we are justified in making the following identifications:

<u>LW Notation</u>	<u>Our Notation</u>
$S_p(\xi_{\ell_p})$	$G^{(0)}(\xi_{\ell}, \varepsilon_p)$
$S'_p(\xi_{\ell_p})$	$G(\xi_{\ell}, \varepsilon_p)$
$G_p(\xi_{\ell_p})$	$M(\xi_{\ell}, \varepsilon_p)$

The identification that is summarized in the table above can now be used for the physical interpretation of various functions and, in particular, the so-called "irreducible self energy operator." This procedure can be done by simply considering the methods and definitions used by LW.

This completes our proof for the validity of our basic equations, (10) and (14), and justifies our assumptions.

CHAPTER IV : THE ENERGY-MOMENTUM REPRESENTATIONS OF THE GREEN'S FUNCTIONS

Introduction.

Hitherto we observed the striking parallelism between the time and the temperature Green's functions. This is further illustrated below where we show that the definitions for the retarded and advanced "time" Green's functions find their natural correspondence in the "temperature" Green's functions. However, when we consider the energy-momentum representations (henceforth denoted by Ep-rep.) of these functions some of the parallelism is lost.

The Ep-reps for the two cases, temperature and time Green's functions, are obtained in two steps. Firstly, the momentum representations are obtained by simply taking the Fourier transforms of the 3-dimensional space dependence of the Heisenberg operators; this procedure is identical in the two cases. Secondly, the energy representations are obtained, again by Fourier-like transformations, with respect to the time and temperature like variable respectively. The resultant expressions are compared and discussed. It is precisely at this point that the parallelism between the time and temperature, which was mentioned above, breaks down. Now, it is possible to split the time Green's in the Ep-rep into two parts (see e.g. Klein and Prange, 1958) in such a way that one part corresponds to a retarded, and the other to an advanced, time Green's function. The question naturally arises whether there is any corresponding

division in the temperature Green's function in the Ep-rep. This point is elaborated in the present chapter. In particular, the case

$\beta \rightarrow \infty$ is examined in detail.

(b) The Ep-reps of the Temperature Green's Function for Finite Temperatures.

From equation (5) Chapter III we have

$$\partial_u \psi(x) = [\psi, H]_-, \text{ where } x = (\underline{r}, iu).$$

The standard solution of this equation is

$$\psi(x) = e^{-uH} \psi(\underline{r}) e^{uH}. \quad (1a)$$

Similarly, one has

$$\psi^+(x) = e^{-uH} \psi^+(\underline{r}) e^{uH}. \quad (1b)$$

The "temperature" Green's function was given (equation (9), Chapter III) by

$$G(x, x') = i \langle T[\psi(x) \psi^+(x')] \rangle.$$

Upon substituting equations (1) into this expression, carrying out a Fourier expansion of the spatial parts, and, finally, demanding that $(\nabla_r + \nabla_r') G = 0$ (i.e. applying the condition for uniform media; see comments of pp. 39, 40), we get

$$G_k(u, u') = - \langle T[a_k(u) a_k(u')] \rangle, \quad (2)$$

where

$$a_k(u) = e^{-uH} a_k e^{uH}, \quad (3a)$$

and

$$a_k^+(u) = e^{-uH} a_k^+ e^{uH}, \quad (3b)$$

and where $G_k(u, u')$ is the Fourier coefficient in the expansion:

$$G(x, x') = \frac{+i}{(2\pi)^3} \int G_k(u, u') e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} d^3k .$$

From (2) and (3) and the cyclic invariance of the trace (which is implied by the angular brackets), we have

$$G_k(u, u') = G_k(u - u') \equiv G_k(v) ; \quad v \equiv u - u' .$$

In fact, (2) can be written more explicitly:

$$G_k(v) = -D \sum_{Na} e^{-\beta E_{N,a} + \alpha N + v E_{N,a}} \langle N, a | a_k e^{-vH} a_k^\dagger | N, a \rangle, \text{ for } v < 0, \quad (4a)$$

$$= D \sum_{Na} e^{-\beta E_{N,a} + \alpha N - v E_{N,a}} \langle N, a | a_k^\dagger e^{vH} a_k | N, a \rangle, \text{ for } v > 0, \quad (4b)$$

where

$$D^{-1} = \sum_{Na} e^{-\beta E_{N,a} + \alpha N}$$

and the states $|N, a\rangle$ are defined by the eigenvalue equation

$H|N, a\rangle = E_{N,a}|N, a\rangle$; here N is the number of particles in the state $|N, a\rangle$ and a stands for all other quantum numbers that are required to specify the state.

In the previous chapter equation (18), and the comments below (20) we used

$$G_K(v) = \frac{1}{\beta} \sum_{-\infty}^{+\infty} G_K(\xi_\ell) e^{-\xi_\ell v} , \quad (5)$$

where we have deleted the label ε_K ; $|u|, |u'| \leq \beta/2$ and $\xi_\ell = \frac{(2\ell + 1)}{\beta} \pi i + \mu$. From (5) we have immediately

$$G_K^A(\xi_\ell) = \int_0^\beta e^{\xi_\ell v} G_K(v) dv \quad \text{for } v > 0 , \quad (6a)$$

and

$$G_K^R(\xi_\ell) = \int_{-\beta}^0 e^{\xi_\ell v} G_K(v) dv \quad \text{for } v < 0 . \quad (6b)$$

Here the superscripts A, R signify the "advanced" ($\nu > 0$), respectively "retarded" ($\nu < 0$), temperature Green's function.

In order to utilize equations (6) it is expedient to rewrite terms occurring in equations (6) in a different form (using a sum over intermediate states)

$$\langle N, a | a_k e^{-\nu H} a_k^+ | N, a \rangle = \sum_{a'} |\langle N, a | a_k | (N+1), a' \rangle|^2 e^{-\nu E_{N+1, a'}}, \quad (7a)$$

$$\langle N, a | a_k^+ e^{+\nu H} a_k | N, a \rangle = \sum_{a'} |\langle N, a | a_k^+ | (N-1), a' \rangle|^2 e^{+\nu E_{N-1, a'}}. \quad (7b)$$

Thus equations (4) can be written as

$$G_k(\nu) = -D \sum_{N, a, a'} e^{-\beta E_{N, a} + \alpha N - \nu (E_{N+1, a'} - E_{N, a})} A_{Na a'}^k, \text{ for } \nu < 0, \quad (8a)$$

$$= D \sum_{N, a, a'} e^{-\beta E_{N, a} + \alpha N - \nu (E_{N, a} - E_{N-1, a'})} B_{Na a'}^k, \text{ for } \nu > 0, \quad (8b)$$

where

$$A_{N,a\dot{a}}^k = B_{N+1,\dot{a}a}^k = |\langle N,a | a_k | (N+1)\dot{a} \rangle|^2 \geq 0, \quad (9a)$$

:

and

$$A_{N-1,\dot{a}\dot{a}}^k = B_{N,a\dot{a}}^k = |\langle N,a | a_k^+ | (N-1)\dot{a} \rangle|^2 \geq 0. \quad (9b)$$

Substituting (8) into (6) we get

$$G_K^A(\mathcal{F}_\ell) = -D \sum_{Na\dot{a}} e^{-\beta E_{N,a} + \alpha N} A_{Na\dot{a}}^k \left[\frac{1 - \exp\{[\mathcal{F}_\ell - (E_{N+1,\dot{a}} - E_{N,a})]\beta\}}{-\mathcal{F}_\ell + (E_{N+1,\dot{a}} - E_{N,a})} \right], \quad (10a)$$

$$G_K^R(\mathcal{F}_\ell) = D \sum_{Na\dot{a}} e^{-\beta E_{N,a} + \alpha N} B_{Na\dot{a}}^k \left[\frac{1 - \exp\{[-\mathcal{F}_\ell - (E_{N,a} - E_{N-1,\dot{a}})]\beta\}}{\mathcal{F}_\ell - (E_{N,a} - E_{N-1,\dot{a}})} \right]. \quad (10b)$$

Upon a little rearrangement these equations take on an identical form:

$$G_K(\mathcal{F}_\ell) = G_K^A(\mathcal{F}_\ell) = G_K^R(\mathcal{F}_\ell) = D \sum_{Na\dot{a}} e^{-\beta E_{N,a} + \alpha N} \left[\frac{B_{Na\dot{a}}^k}{\mathcal{F}_\ell - (E_{N,a} - E_{N-1,\dot{a}})} + \frac{A_{Na\dot{a}}^k}{\mathcal{F}_\ell - (E_{N+1,\dot{a}} - E_{N,a})} \right]. \quad (11)$$

Because of the properties of the operators a_k and a_k^\dagger this equation can be written in a more compact form:

$$G_k(\mathcal{E}_\lambda) = D \sum_{n,m} \frac{\exp[-\beta(E_n - \mu N_n)] + \exp[-\beta(E_m - \mu N_m)]}{\mathcal{E}_\lambda - (E_n - E_m)} B_{m,n}^k, \quad (12)$$

where m, n now run over all states and over all particle numbers, and

$$B_{m,n}^k = |\langle m | a_k | n \rangle|^2 \geq 0. \quad (13)$$

The fact that, in the Ep-rep, the "retarded" and "advanced" temperature Green's functions are identical is a reminder that the analogy between the temperature variable and time is not exact. This will become clear when we discuss the retarded and advanced time Green's functions in the next section.

(c) The Ep-rep of the Time Green's Function for Finite Temperatures.

We now summarize the salient features of the Ep-rep for the time Green's functions. Proceeding in much the same way as was done for the temperature variable in the previous section we get

$$\psi(x) = e^{itH} \psi(r) e^{-itH}, \quad (14a)$$

$$\psi^+(x) = e^{itH} \psi^+(r) e^{-itH}. \quad (14b)$$

The Green's function in the momentum representation ($\tau = t - t'$)

$$G_K(\tau) = -D \sum_{naa} e^{-\beta E_{N,a} + \alpha N - i\tau(E_{N+1,a} - E_{N,a})} A_{naa}^k \text{ for } \tau > 0, \quad (15a)$$

$$= D \sum_{naa} e^{-\beta E_{N,a} + \alpha N - i\tau(E_{N,a} - E_{N-1,a})} B_{naa}^k \text{ for } \tau < 0. \quad (15b)$$

Comparing equations (12) with (15) we see immediately that $G_K(\tau)$ ($\tau \geq 0$) is the analytic continuation of $G_K(v)$ ($v \geq 0$): $v \rightarrow -i\tau$

(cf. Fradkin, 1959). Thus, given $G_K(\nu)$, we can immediately write down the corresponding $G_K(\lambda)$. This is not surprising in view of the parallel formulations which we have presented in Chapter III for the temperature and time Green's functions.

We now define the following functions:

$$\begin{aligned} G_K^{(a)}(\lambda) &= i G_K(\lambda) \quad \lambda > 0, \\ &= 0 \quad \lambda \leq 0; \\ G_K^{(r)}(\lambda) &= 0 \quad \lambda > 0, \\ &= i G_K(\lambda) \quad \lambda \leq 0. \end{aligned} \quad (16)$$

The function $G_K^{(a)}(\lambda)$, respectively $G_K^{(r)}(\lambda)$, can be readily expanded into its Fourier decomposition in the lower, respectively upper, half of the \mathbb{Z} plane; the result is

$$G_K^{(a)}(z) = D \sum_{N,a} e^{-\beta E_{N,a} + \alpha N} \frac{1}{E_{N+1,a} - E_{N,a} - z} A_{Na}^K, \quad \text{Im } z < 0, \quad (17a)$$

$$G_K^{(r)}(z) = -D \sum_{N,a} e^{-\beta E_{N,a} + \alpha N} \frac{1}{E_{N,a} - E_{N-1,a} - z} B_{Na}^K, \quad \text{Im } z > 0, \quad (17b)$$

where

$$G^{(a,r)}(z) = \int_{-\infty}^{+\infty} G_K^{(a,r)}(\lambda) \exp[-iz\lambda] d\lambda,$$

and where the notation of the previous section has been employed.

The singularities of these expressions, when regarded as functions of the complex variable \bar{z} , occur only along the real axis. Furthermore in the limit of an infinitely large system, as is considered here, each point is usually a limit of poles and, therefore, the real axis is in general a branch line (cf. Luttinger, 1961). Hence $G_K^{(a)}(\bar{z})$, respectively $G_K^{(r)}(\bar{z})$, can be analytically continued into the whole lower, respectively upper, plane.

(d) Comparison Between the Ep-rep of the Temperature and Time Green's Functions at Finite Temperatures.

We now split the Ep-rep of the temperature Green's function (equation (11)) into two parts: $G_K(\zeta_x) =$

$G_K^+(\zeta_x) + G_K^-(\zeta_x)$; where

$$G_K^+(\zeta_x) = -D \sum_{Naa'} e^{-\beta E_{N,a} + \alpha N} \frac{A_{Na a'}^K}{E_{N+1,a'} - E_{N,a} - \zeta_x}, \quad (18a)$$

and

$$G_K^-(\zeta_x) = -D \sum_{Naa'} e^{-\beta E_{N,a} + \alpha N} \frac{B_{Na a'}^K}{E_{N,a} - E_{N-1,a'} - \zeta_x}. \quad (18b)$$

In the above expressions we have used equation (9a) and the assumption that the summations over a and a' run over the same quantum states. The last assumption is based on the fact that, physically, we expect the N particle and $N \pm 1$ particle systems to be characterized by the same quantum numbers. As in the case of the Ep-rep of the time Green's functions, these expressions (regarded as functions of a complex variable ζ ($\zeta_x \rightarrow \zeta$)) are analytic everywhere except on the real axis.

Comparing (18a) with (17a) and likewise (18b) with (17b), we observe that these expressions are simply related in the upper,

respectively lower, planes. From these observations we can readily see that $i G_A^+(\xi)$ represents the Ep-rep of the advanced Green's function in the upper plane, and $-i G_h^-(\xi)$ the retarded Green's function in the lower plane. (The factor i arises from the definition (16)). It is difficult, and perhaps impossible, to extend the identification to the whole plane at finite temperatures because, in this case, the whole real axis could be a branch line in the complex plane. The latter is due to the presence, in general, of continuous spectra of energies and of degeneracies at the various excited states. However, in the zero temperature limit one can extend the identification to the whole complex plane. This is discussed in the next section.

(e) Ep-rep of the Green's Functions in the Limit of Zero Temperature.

In the limit of zero temperature, equations (15) reduce to

$$G_K(t) = - \sum_n e^{-it(E_n - E_{N,0})} B_{n,0}^K = \langle N,0 | a_K e^{-i(H-E_{N,0})t} a_K^\dagger | N,0 \rangle, \text{ for } t > 0, (19a)$$

$$= \sum_n e^{-it(E_{N,0} - E_n)} B_{0,n}^K = \langle N,0 | a_K^\dagger e^{i(H-E_{N,0})t} a_K | N,0 \rangle, \text{ for } t < 0, (19b)$$

where we consider an N particle system whose ground state is given by $|N,0\rangle$ and its ground state energy by $E_{N,0}$. The sum over n runs over all states and all particle numbers; $n=0$ denotes the ground state which is assumed to be non-degenerate.

That equations (19) are correct is clear inasmuch as all other terms in (15) are exponentially smaller in the $\beta \rightarrow \infty$ limit as compared to the terms given by (19). Proceeding as in section (c) we get

$$G_K^{(a)}(z) = \langle N,0 | a_K \frac{1}{H - E_{N,0} - z} a_K^\dagger | N,0 \rangle, \text{ for } \text{Im } z > 0 \quad (20a)$$

and

$$G_K^{(r)}(z) = - \langle N,0 | a_K^+ \frac{1}{E_{N,0} - H - z} a_K | N,0 \rangle \quad \text{for } \text{Im } z < 0 \quad (20b)$$

The zero temperature limits for the "temperature" Green's functions in the Ep-rep (cf. Luttinger, 1961) are given by

$$G_K^+(\mathcal{E}) = - \langle N,0 | a_K \frac{1}{H - E_{N,0} - \mathcal{E}} a_K^+ | N,0 \rangle, \quad (21a)$$

and

$$G_K^-(\mathcal{E}) = - \langle N,0 | a_K^+ \frac{1}{E_{N,0} - H - \mathcal{E}} a_K | N,0 \rangle. \quad (21b)$$

With regard to the derivation of equations (21), we merely remark here that in the limit $\beta \rightarrow \infty$, $\mathcal{E}_\ell \equiv \frac{\pi i}{\beta}(2\ell+1) + \mu$ becomes a continuous variable, and hence equation (21b) becomes an integral transform equation. The details of the derivation are given in Luttinger (1961) and in Luttinger and Ward (1960).

If we regard $G_{\kappa}^{\pm}(\xi)$ as a function of the complex variable ξ we note immediately that the function is analytic everywhere except on the real axis. But in the case of interest now (i.e. $\beta \rightarrow \infty$) we shall show that both functions are analytic also on infinite segments of the real axis. Physically this can be seen as follows: - the chemical potential for an N particle system can be interpreted as the change in the energy of the system upon removal of one particle at fixed β and volume: $\mu_N = E_{N,0} - E_{N-1,0}$, where both energies are ground state energies. Furthermore, for large N we expect (e.g. Pines, 1961) that $\mu_N \simeq \mu_{N+1} \simeq \mu_{N-1}$, which we denote simply by μ . The correction to this equation is of the order of $1/N$ and, in our case, can be neglected. With this equality we can write the energy differences occurring in equations (21) and (20) as follows (cf. Pines, 1961):

$$E_{N,0} - E_{N-1,a} = (E_{N,0} - E_{N-1,0}) - (E_{N-1,a} - E_{N-1,0}) \leq \mu,$$

$$E_{N+1,a} - E_{N,0} = (E_{N+1,a} - E_{N+1,0}) + (E_{N+1,0} - E_{N,0}) \geq \mu,$$

and thus equations (21a) and (21b) are regular on the real axis in the region $-\infty$ to $+\infty$ ($-\infty$ to $+\infty$). This clearly holds for equations (20) as well. Hence we conclude that in the zero temperature limit, $iG_K^+(\xi)$ represents the Ep-rep of the advanced Green's function in the whole complex ξ plane, and similarly $-iG_K^-(\xi)$ represents the retarded Green's function. The reason that this identification is possible in the zero temperature limit is, clearly, the possibility of analytic continuation of $G_K^{(a,r)}(z)$ into the upper and lower planes, since in this limit ($\beta \rightarrow \infty$) the real axis is not a branch line from $-\infty$ to $+\infty$.

(f) Spectral Representation for the Green's Function in the Zero Temperature Limit.

For an infinite system the energy spectrum is expected to form a continuum, i.e. the poles of equation (25) come close together and the real axis is, in general, a branch line (Luttinger, 1961). For the case of very closely spaced energy levels it is appropriate to convert the summations into integrals. This is done customarily (e.g. Pines, (1961), Luttinger, (1961)) by introducing the spectral functions

$$A(k, \omega) = \sum_n A_{n,0}^k \delta(\omega - \omega_{n0}) ,$$

$$B(k, \omega) = \sum_n B_{n,0}^k \delta(\omega - \omega_{n0}) .$$

Here $\omega_{n0} = E_n - E_{N,0}$ and $A_{n,0}^k$ & $B_{n,0}^k$ are defined by equations (9) and (13). With these definitions the formulae (25) can be written as

$$G_k^+(\xi) = - \int_{-\infty}^{\mu} \frac{A(k, \omega) d\omega}{\omega - \xi} ,$$

$$G_k^-(\xi) = - \int_{\mu}^{\infty} \frac{B(k, \omega) d\omega}{\omega - \xi} ,$$

where ω plays the rôle of the energy differences in equations (25). The association of the time and temperature E_p -reps will, clearly, go through in much the same way as before.

CHAPTER V: CONCLUSIONS AND OUTLOOK

We begin this chapter with a short historical survey of the subject. This survey, which one might normally have expected in the introductory chapter, has been deferred for presentation here because the literature in this field is relatively recent and the problem of discriminating between the contributions of various authors is more simply handled after the general methods have been discussed.

The temperature-dependent many body theory aims, as does its zero temperature counterpart, to obtain expressions for the energy spectrum and the mean density of a system when the mutual interaction of the particles is given.

The $T = 0$ case was solved, in a certain sense, by Brueckner and his collaborators (references are given in Goldstone's paper); however, their approach was ambiguous and difficult. Goldstone (1957) was the first to attack the problem using the time-dependent techniques associated with Feynman diagrams borrowed from field theory. His approach was further simplified by Bloch (1958). The modern approach, wherein extensive use is made of the Green's function formalism, was initiated by Klein and Prange (1958); these authors extended their considerations to the $T \neq 0$ case.

The theory of finite temperature MBP is intrinsically more complex than the $T = 0$ case because of the inherent requirement for statistical averaging; the theory cannot, at present, point to a success comparable to the theory of Brueckner and Goldstone for zero temperature. Nonetheless, some important achievements have been made: (a) It is now possible to obtain, in the $T \rightarrow 0$ limit, the Goldstone expression for the ground state, without recourse to time-dependent theory (the Green's function formalism can be considered here as a concise formulation for the time-dependent approach). This point was clarified by Luttinger and Ward (1960), who considered a perturbation expansion wherein a temperature-like variable (denoted by u in this thesis) played the role of time. (b) The semi-phenomenological theory for Fermi liquids proposed by Landau (1957) was put on a firm theoretical footing by Klein (1961b), who used the "time"-Green's function. This theory was also derived from first principles by Balian, Bloch and DeDominicis (1961) who used the "generalized Wick contractions" first devised by Bloch and DeDominicis in 1958; this method also employs the u variable explicitly. (c) Luttinger (1960) obtained a temperature-dependent extension for the Hartree-Fock approximation which reduces (see qualifying remarks in Thouless, 1961) to the standard Hartree-Fock expression in the zero temperature limit. This approach once again uses the u variable as an auxiliary tool for the discussion.

AGD (1959) and Fradkin (1959) re-examined an approach to the T-dependent MBP which had been suggested originally by Matzubara (1955). These authors extended Matzubara's u-variable approach and pointed out an interesting parallelism between the rôles played by the time and the u-variable in the theory. For example, AGD considered the retarded "time" Green's function and showed it to be the analytic continuation of the "temperature" Green's function. Fradkin in a relativistic formulation of the problem considered the "temperature" Green's function in some detail and outlined a scheme for the determination of the energy spectrum of a system in terms of it.

As may be clear to the reader, the main interest in this thesis, apart from an elaboration of the Green's function itself, has been the study of the curious and suggestive parallelism between the rôles played by the time and the u-variable in the T-dependent MBP. We outlined the conventional theory of the Green's function (i.e. the "time" Green's function) as it has recently been formulated for the MBP. Then a new variable, τ , was introduced into the theory; τ was shown to play the two rôles, viz. time and u in the conventional theory. The purpose of this formalism was to make the parallelism between the u and t variables manifest; but it also enabled us to rederive results which had been hitherto obtained by Luttinger and Ward (without recourse to the Green's function formalism). It is at once evident that our derivation is simpler and also appreciably shorter.

The striking parallelism between the roles of the time and the temperature variable in the τ formalism (and, of course, in general) is, as one would expect, not complete: for one thing the u -variable occurs with an imaginary argument in τ while the time occurs with a real argument; also there is the fact that the variable u is confined to a finite region ($|u| \leq \beta/2$) while t is unbounded. These differences are manifested most clearly in the Ep-rep where we showed that while one is able in the usual way to distinguish between the "retarded" and "advanced" time Green's functions, this distinction is not present for the "temperature" Green's functions. However, we have shown that in the Ep-rep the "temperature" Green's function can be split into two parts, one of which can be associated most naturally with the advanced and the other with the retarded time Green's function.

It is our hope that the τ formalism can be used to investigate a further-going connection between the t and u variables. We have also speculated briefly on the possibility that the introduction of non-diagonal elements in the resultant Green's function might correspond to terms describing phenomena associated with the time development of non-equilibrium conditions*;

* e.g. Professor Y. Nambu in a private discussion suggested that insertion of some appropriately chosen, non-diagonal terms into our Green's function might lead to a new formulation of the problem of thermal conductivity.

however, our predilection lies in further investigation of the above-mentioned analogy between time and the temperature variable u . In particular, in continuing the investigation, the splitting of the temperature Green's function in the Ep-rep into "retarded" and "advanced" parts will be examined in other representations to see whether a deeper significance to the analogy between the u variable and time can be uncovered.

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APPENDIX A: VERIFICATIONS OF SOME FORMULAE GIVEN IN CHAPTER II

The proofs and discussions given below lean heavily on the lucid review article by Kato et. al., (1960).

(i) Source terms in the Lagrangian density.

Let us consider a closed system with no external sources. Since the Lagrangian will then have no explicit dependence on the time, the resultant Hamiltonian will possess a spectrum of stationary eigenvectors which we label by $\{m_i\}$. We now add the following terms to this Lagrangian,

$$-J^+(x)\psi(x) - \psi^+(x)J(x),$$

where $J(x)$ is an infinitesimal external^{source}, which is independent of time for $t \ll t'$ and for $t \gg t'$. As an example, consider $J(x)$ to have a step function time-dependence: zero, say, for

$$-\infty < t \leq t', \quad \text{non-zero but infinitesimal for}$$

$$t' < t < \infty. \quad \text{The Hamiltonian resulting from this}$$

modified Lagrangian will not have a stationary spectrum of eigenvectors. Nonetheless, we expect that for $t \ll t'$ it does have the spectrum $\{m_i\}$, while for $t \gg t'$ it has a new spectrum which we label $\{m_f\}$; further, in view of the source being infinitesimal, we expect and assume, that there exists a one to one correspondence between the spectra $\{m_i\}$ and

$\{m_f\}$ such that, as we set the source equal to zero, there exists an m_f which becomes identical with m_i for every m_i in the spectrum. More succinctly, we demand for physical Hamiltonians:

$$\exists m_f \forall m_i, \exists m_f \rightarrow m_i \text{ as } J \rightarrow 0. \quad (\text{A1})$$

(ii) Summation over initial and final spectra (equation (8)):

With the correspondence (A1) of m_f to m_i for every m_i , the meaning of the summation defining equation (8b) can be made explicit by writing

$$\rho \equiv \sum_{m_i} |m_i\rangle \omega_{m_i} \langle m_f| \quad (\text{A2})$$

where ω_{m_i} is the weight factor specified by equation (8).

(iii) The functional derivative: $\delta \text{Tr. } \{P\} / \delta J(x)$.

Let us now vary the Lagrangian by letting

$J(x) \rightarrow J(x) + \varepsilon f(x)$, where it is convenient to assume that ε is such as to make $\varepsilon f(x)$ an infinitesimal of higher order than $J(x)$, and $f(x)$ is an arbitrary function in the sense of equation (1); for convenience we assume it to be localized in time around t' . We label the independent variations of $J(x)$ and $J^\dagger(x)$ by δ and δ_+ , respectively; e.g. $\delta_+ \mathcal{L}$ reads: the variation in \mathcal{L} due to the variation in $J^\dagger(x)$. With these preliminary remarks we can take

the functional derivative of $\text{Tr.} \rho$ with respect to the source functions as follows:

$$\delta \text{Tr.} \rho = \delta \sum_{m_i} \omega_{m_i} \langle m_f | m_i \rangle ,$$

which, through Schwinger's principle, becomes

$$\delta \text{Tr.} \rho = -i \sum_{m_i} \omega_{m_i} \langle m_f | \int d^4x \psi^\dagger(x) \varepsilon f(x) | m_i \rangle . \quad (\text{A } 3)$$

On the other hand,

$$\delta \text{Tr.} \rho = \text{Tr.} \rho \Big|_{J+\varepsilon f} - \text{Tr.} \rho \Big|_J \equiv \int \frac{\delta \text{Tr.} \rho}{\delta J(x)} \varepsilon f(x) d^4x \quad (\text{A } 4)$$

in the limit $\varepsilon \rightarrow 0$. The last identity follows from the definition of functional differentiation (equation (1)).

Comparing (A3) with (A4) we get, since $f(x)$ is arbitrary,

$$\frac{\delta \text{Tr.} \rho}{\delta J(x)} = -i \sum_{m_i} \omega_{m_i} \langle m_f | \psi^\dagger(x) | m_i \rangle . \quad (\text{A } 5)$$

(iv) The functional derivative:

Since the source functions must be spinors for the Lagrangian to be scalar, they anticommute with other spinors such as $Q(x)$. Hence, in considering the matrix element $\langle m_f | Q(x) | m_i \rangle$, we get two distinct possibilities depending on whether the time associated with Q (labelled by t) is earlier or later than the time of variation of the source (t'). For $t > t'$ we choose a time t_1 such that $t > t_1 > t'$, so that at t_1 we have once again a spectrum satisfying the stipulation (A1). We label this spectrum by $\{m(t_1)\}$; hence, we have immediately:

$$\frac{\delta}{\delta J(x)} \langle m_f | Q(x) | m_i \rangle = \sum_{m(t_1)} \langle m_f | Q(x) | m(t_1) \rangle \frac{\delta \langle m(t_1) | m_i \rangle}{\delta J(x)},$$

where $|m(t_1)\rangle$ plays a role of $|\alpha\rangle$ in Schwinger's dynamical principle. The same discussion leading to (A5) now leads to:

$$\frac{\delta}{\delta J(x)} \langle m_f | Q(x) | m_i \rangle = -i \langle m_f | Q(x) \psi^\dagger(x) | m_i \rangle. \quad (\text{A6a})$$

An analogous procedure for $t' > t$ leads to:

$$\frac{\delta}{\delta J(x')} \langle m_f | Q(x) | m_i \rangle = i \langle m_f | \psi^\dagger(x') Q(x) | m_i \rangle, \quad (\text{A6b})$$

where the sign change is due to the anticommutativity of the source with the spinor $Q(x)$. Equations (A6) can be written as:

$$\frac{\delta}{\delta J(x')} \langle m_f | Q(x) | m_i \rangle = -i \langle m_f | T[Q(x) \psi^\dagger(x')] | m_i \rangle, \quad (\text{A7})$$

where T is the chronological operator of Wick.

(v) The statistical Green's function.

The Green's function of our problem was defined by:

$$G(x, x') \equiv - \frac{\delta \langle \psi \rangle}{\delta J(x')} \bigg|_{J=0}$$

where the expectation values, signified by the angular bracket, is defined in equation (8):

$$\langle \psi(x) \rangle \equiv \frac{\sum_{m_i} \omega_{m_i} \langle m_f | \psi(x) | m_i \rangle}{\sum_{m_i} \omega_{m_i} \langle m_f | m_i \rangle}$$

Hence,

$$\frac{\delta \langle \psi(x) \rangle}{\delta J(x)} = \frac{\frac{\delta}{\delta J(x)} \sum_{m_i} \omega_{m_i} \langle m_f | \psi(x) | m_i \rangle}{\text{Tr. } \rho} - \frac{\langle \psi \rangle \frac{\delta}{\delta J(x)} \text{Tr. } \rho}{\text{Tr. } \rho};$$

which upon application of (A7) and (A3) reads:

$$\frac{\delta \langle \psi(x) \rangle}{\delta J(x)} = -i \langle T[\psi(x) \psi^\dagger(x')] \rangle + i \langle \psi(x) \rangle \langle \psi^\dagger(x') \rangle,$$

where now $\psi(x)$ plays the role of $Q(x)$. If we now take the limit $m_f \rightarrow m_i$ (by letting $J \rightarrow 0$), the second term vanishes and we get finally:

$$-G(x, x') \equiv \left. \frac{\delta \langle \psi(x) \rangle}{\delta J(x)} \right|_{J=0} = -i \langle T[\psi(x) \psi^\dagger(x')] \rangle,$$

in agreement with equation (9).

(vi) The Schwinger identity.

The Schwinger identity states the following operator relation:

$$\partial_t T[\psi(x) \psi^\dagger(x')] = T[\partial_t \psi(x) \psi^\dagger(x')] + \delta^4(x - x').$$

The proof can be obtained in the following way:

$$T[\psi(x) \psi^\dagger(x')] \equiv \theta(t-t') \psi(x) \psi^\dagger(x') - \theta(t'-t) \psi^\dagger(x') \psi(x), \quad (\text{A8})$$

where $\theta(x)$ is the usual step function

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}.$$

Differentiating (A8) partially with respect to the time t , we get:

$$\begin{aligned} \partial_t T[\psi(x) \psi^\dagger(x')] &= \delta(t-t') \psi(x) \psi^\dagger(x') + \theta(t-t') \partial_t \psi(x) \psi^\dagger(x') \\ &\quad + \delta(t-t') \psi^\dagger(x') \psi(x) - \theta(t'-t) \psi^\dagger(x') \psi(x), \end{aligned} \quad (\text{A9})$$

where we have used the well-known formula: $\partial_x \theta(x) = \delta(x)$.

Using the commutation relation of ψ and ψ^\dagger we can combine the first and third terms in (A9), while the second and fourth terms can be combined using the definition of the chronological operator T . These combinations result in the Schwinger identity (equation (13)).

APPENDIX B. COMPLETENESS OF THE SET $\{ \phi_{p,\ell} \}$.

The set $\phi_{p,\ell}$ was defined by (equation (20),

Chapter III):

$$\phi_{p,\ell}(x,v) = \frac{i}{\beta V} \exp \left[i \underline{p} \cdot \underline{r} - \mathcal{E}_\ell v \right].$$

The spatial factor is the familiar Fourier one, and as such, requires no further comment. The factor containing v and involving

$$\mathcal{E}_\ell \equiv \frac{2\ell+1}{\beta} \pi i + \mu \quad \text{possesses certain symmetry properties;}$$

hence, it is necessary to investigate whether this set is sufficiently complete for our purposes. We consider the spatial Fourier component of $G(x, \dot{x})$ and show that it satisfies the requirements which warrant an expansion in terms of

$$\frac{1}{\beta} \exp \left[- \mathcal{E}_\ell v \right].$$

We had,

$$G_p(u, \dot{u}) = - \left\langle T \left[a_p(u) a_p^\dagger(\dot{u}) \right] \right\rangle.$$

Written more fully this equation reads ($v \equiv u - \dot{u}$):

$$G_p(v) = -D \text{Tr.} e^{-\beta H + \alpha N} e^{-vH} a_p e^{vH} a_p^+ \quad \text{for } v < 0, \quad (\text{B1a})$$

$$G_p(v) = D \text{Tr.} e^{-\beta H + \alpha N} e^{vH} a_p^+ e^{-vH} a_p \quad \text{for } v > 0, \quad (\text{B1b})$$

(see equations (3), (4) of Chapter IV).

Now consider the expression (Luttinger, 1961; Bloch, 1960);

$$g_p \equiv G_p(v) \exp\left\{\left[\frac{\pi i}{\beta} + \mu\right]\beta\right\}.$$

This expression has the following properties:

$$g_p(v') \equiv g_p(v - \beta) = g_p(v) \quad \text{for } v' < 0, v > 0,$$

$$g_p(v'') \equiv g_p(v + \beta) = g_p(v) \quad \text{for } v'' > 0, v < 0.$$

These properties can be established from equations (B1), by using the cyclic invariance of the trace and the following properties of the operators a_k and a_k^+ :

$$a_k(N-1) = N a_k ,$$

$$a_k^+(N+1) = N a_k^+ .$$

This symmetry property of $g_p(v)$ is summarized by the statement: $g_p(v)$ is periodic with a period β in the interval $(-\beta, \beta)$. Since only this interval enters into our calculation, we can assume $g_p(v)$ to be periodic everywhere. This observation implies that $g_p(v)$ can be expanded in the Fourier series

$$g_p(v) = \frac{1}{\beta} \sum_{\ell} G_p(\xi_{\ell}) \exp\left[\left(-\frac{2\ell}{\beta} \pi i\right) v\right],$$

and hence

$$G_p(v) = \frac{1}{\beta} \sum_{\ell} G_p(\xi_{\ell}) \exp[-\xi_{\ell} v] .$$

Clearly a similar proof will go through for $G_p^{(0)}(\vec{r})$, the non-interacting Green's function.

Now equation (14b) of Chapter III can be expressed by:

$$G = \frac{1}{[G^{(0)}]^{-1} - M},$$

which, in turn, can be written in terms of the inverse Green's function:

$$-M = [G]^{-1} - [G^{(0)}]^{-1}.$$

Naturally, if G and $G^{(0)}$ can be expressed in terms of $\phi_{p,l}$ so can M .

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